Non-Standard Models of Arithmetic

Theorem: The arithmetical truth $\forall x\ \forall y\ x + y = y + x$ is not a FOL consequence of PA.

Proof: For $\forall x\ \forall y\ x + y = y + x$ to be a FOL consequence of PA, it needs to be true that for all possible interpretations that make PA true, the interpretation makes $\forall x\ \forall y\ x + y = y + x$ true as well. We will show, however, that there exists an interpretation in which PA is true, but in which $\forall x\ \forall y\ x + y = y + x$ is false. Here it is:

Interpretation: N^{+2}

Domain: The set of objects $\{q_0, q_1, d_0, d_1, d_2, \ldots\}$ (in other words, an infinite number of objects d_i, together with two additional objects q_0 and q_1).

$N^{+2}(\mathbf{0}) = d_0$

$N^{+2}(\mathbf{s}) = s^{+2}$, where:

$s^{+2}(q_0) = q_0$

$s^{+2}(q_1) = q_1$

$s^{+2}(d_i) = d_{i+1}$

$N^{+2}(\mathbf{+}) = +^{+2}$, where $+^{+2}$ is given by following table:

$+^{+2}$	y			
	$x +^{+2} y$	q_0	q_1	d_j
	q_0	q_0	q_0	q_0
x	q_1	q_0	q_0	q_1
	d_i	q_0	q_0	d_{i+j}

$N^{+2}(\mathbf{x}) = x^{+2}$, where x^{+2} is given by following table:

x^{+2}	y				
	$x\ x^{+2}\ y$	q_0	q_1	d_0	d_j
	q_0	q_0	q_0	d_0	q_0
x	q_1	q_0	q_0	d_0	q_0
	d_0	q_0	q_0	d_0	d_0
	d_i	q_0	q_0	d_0	d_{i*j}

It is easily verified that on this interpretation, all axioms of PA are true, and yet $\forall x\ \forall y\ x + y = y + x$ is false. In fact, many other arithmetical truths no longer hold, such as $\forall x\ x + s(0) = s(0) + x$ and $\forall x\ \forall y\ x \times y = y \times x$. Also, by defining Even and Odd in the usual way, q1 is neither Even or Odd!